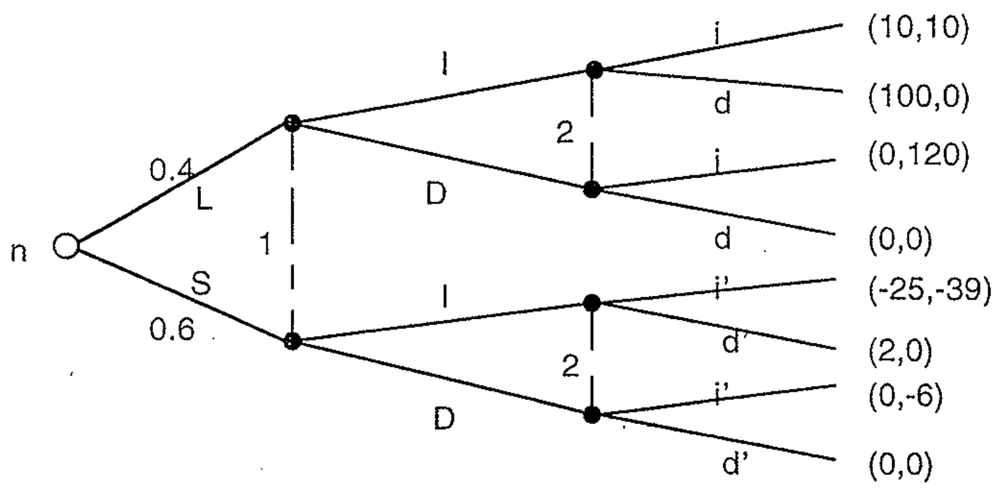
## Chapter VI. Introduction to Game Theory

### 6.1. Basic Definitions

1. In order to introduce the formal concepts of a game in extensive form, we first examine an example.

**Example 1**. The manager of Firm 1 is thinking about introducing a product A. The fixed cost for him to set up production is \$40,000. At the same time, the manager of Firm 2 is thinking about introducing a product B. The fixed cost for her to set up production is \$60,000. There is some uncertainty in the market situation. The total demand of these two products will be either 20,000 or 6,000 units, each with a probability of 0.4 and a probability of 0.6, respectively; and in either situation, each firm gets a one-half market share. The products can be sold in a price \$12 for each unit in monopoly situation, or \$10 in competition situation; raising the price will cause sales to plummet, whereas lowering the price will not increase demand appreciably. The marginal production costs are \$5 for Firm 1, and \$3 for Firm 2. Firm 1 cannot foresee the market situation (with large or small demand), but Firm 2 can predict it because she has already completed a market survey.

[Analysis]. We can describe this story by a "game tree", assuming the market situation is determined by nature.




We now have "a game in extensive form". The managers of the two firms are two players, which will be denoted 1 and 2, respectively. The initial node of the game tree belongs to he once did and what he once knew. Since every finite game (P and S being finite sets) in extensive form with perfect recall can be translated into strategic form, the definition of Nash equilibrium is also good for this kind of extensive form game.

9. Exercise. Find a Nash equilibrium for the game in Example 1.

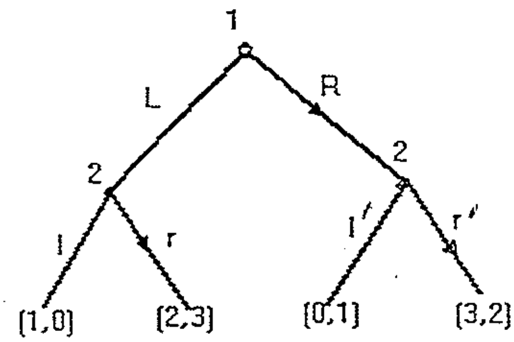
10. Exercise. Construct the game tree for the game described below, and translate it into bimatrix form:

> THE HIDDEN PEARL. There are two dark boxes. Player 1 hides a pearl in one of them, then Player 2, not knowing which box contains the pearl, peeks into one of them. If the pearl is in Box 1 and she looks there, she sees it with probability 0.5; if it is in Box 2 and she looks there, she sees it with probability 0.3. 1 must pay $5 to 2 if she finds the pearl, otherwise 2 must pay $2 to 1.

### 6.2. Computation of Nash Equilibria in Pure Strategies

1. A game may or may not have a Nash equilibrium in pure strategies. When a game [P,S,$\pi$] does have some NEs in pure strategies and |P| and |S| are not very large, we can discover an NE by checking the "best response relationship".

2. **Example 1**. Translate the extensive form game depicted below into strategic form. Then compute all the NEs.




[Analysis]. Player 1 has two pure strategies: *L*, *R*. Player 2 has four pure strategies: *l-l*, *l-r*, *r-l*, *r-r*. Note here that, on making her decision, Player 2 can observe what Player 1 has chosen. Therefore a startegy of her consists of two contingent choices, one in respond to the choice *L* of Player 1, another to the choice of *R*. The bimatrix form of this game is

| | | Player 2 | | | |
|---|---|---|---|---|---|
| | | *l-l* | *l-r* | *r-l* | *r-r* |
| Player 1 | *L* | 1, 0 | 1, 0 | 2, 3 | 2, 3 |
| | *R* | 0, 1 | 3, 2 | 0, 1 | 3, 2 |

Now it is easy to check that there are three NEs in pure strategies: $\langle L, r\text{-}l\rangle$, $\langle R, l\text{-}r\rangle$ and $\langle R, r\text{-}r\rangle$.

3. When |P| or |S| is large, it is time-consuming to check the best response relationship. Under some circumstances, we may simplify the normal form game by deleting the "dominated strategies".

A strategy $s^i_1$ of Player i is said to be weakly dominated by another strategy $s^i_2$ of his if, for any strategies $s^{-i} = \langle s^1, ..., s^{i-1}, ,s^{i+1}, ..., s^I)\rangle$ played by the other players, we have

$$\pi^i(\langle s^i_1, s^{-i}\rangle) \leq \pi^i(\langle s^i_2, s^{-i}\rangle)$$

If, in addition, the equality sign in the above inequality never holds, then $s^i_1$ is said to be strictly dominated by $s^i_2$. A strategy $s^i$ of Player i is said to be a strictly (weakly) dominant strategy of his, if any other strartegy of i is strictly (weakly) dominated by $s^i$.

4. **Proposition 1**. Assume that game G' is the same as game G except that one strictly dominated strategy $t^i$ of Player i in G has been eliminated in G'. Then G' and G has the same set of Nash equilibria.

Proof. Let $\langle s^{*i}, s^{*-i}\rangle$ be an NE of G. Obviously $s^{*i} \neq t^i$. Thus $s^{*i}$ remains to be a strategy of i's in G' and remains to be a best response of i's to $s^{*-i}$. The choice of every player $j \neq i$ in $\langle s^{*i}, s^{*-i}\rangle$ in playing G' is under precisely the same situation as his choice in $\langle s^{*i}, s^{*-i}\rangle$ in playing G, and it thus remains to be a best response to the choices of the others'. Therefore $\langle s^{*i}, s^{*-i}\rangle$ is an NE of G'.

Conversely, assume that $\langle s^{*i}, s^{*-i}\rangle$ is an NE of G'. When $\langle s^{*i}, s^{*-i}\rangle$ is regarded as a strategy profile of G, every player $j \pi i$ has chosen her best response to the choices of the others'. What we need to argue is that $s^{*i}$ remains to be a best response of i's to $s^{*-i}$ in G. In fact, if $s^{*i}$ were not i's best response to $s^{*-i}$ in G, then i's only best response to $s^{*-i}$ would have to be $t^i$, which is impossible since $t^i$ is a strictly dominated strategy.

Note that according to this proposition, to compute a Nash equilibrium in pure strategies for a game G, we can "simplify" G by iteratively eliminating the strictly dominated strategies until we obtain a game G' with no strictly dominated strategy for any player. Then the set of NEs for G' is the set of NEs for G.

5. **Example 2**. (The Prisoners' Dilemma) Two thieves have been caught and are facing jail sentences. They are placed in separate cells where there is no possiblility of them being able to communicate with each other. Separately, each prisoner is asked to confess. If one confesses but the other does not, then the one who confesses will be sentenced to jail for 1 year, and the other for 10 years; if both confess, each will be sentenced to jail for three years; and, if no one confesses, each will be sentenced to jail for 2 years. All above are known to both of them. But neither of them will be told what the other has chosen to do.

[Analysis]. Each prisoner has two pure strategies: confess (C), do not confess (D). Regarding the number of years in jail as disutility, we can construct the payoff bimatrix:

| | | Prisoner 2 C | Prisoner 2 D |
|---|---|---|---|
| Prisoner 1 | C | -3, -3 | -1, -10 |
| | D | -10, -1 | -2, -2 |

It is easy to see that for each prisoner, D is a strictly dominated strategy. Thus the only Nash equilibrium of this game is <C, C> with each prisoner being sentenced for 3 years.

From this example we also observe that a Nash equilibrium need not be associated with a Pareto optimal outcome.

6. Exercise. Given a game G. Assume that G' is obtained from G by eliminating a weakly dominated strategy of Player i's in G. Show that any Nash equilibrium for G' is also a Nash equilibrium for G, but it may be not true conversely.

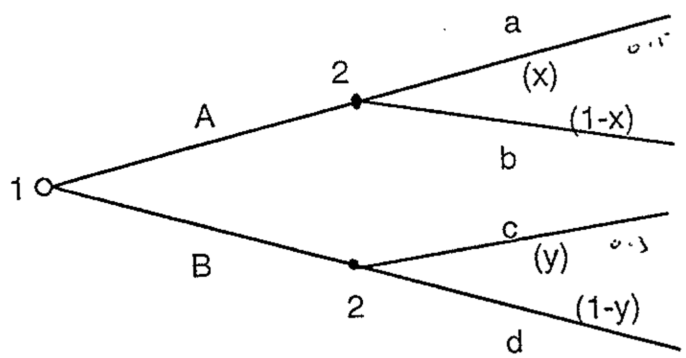
7. Exercise. Compute all the NEs in pure strategies for the following bimatrix games.

### 6.3. Existence of Nash Equilibrium

1. A game may have no Nash equilibrium in pure strategies. To guarantee the existence of an equilibrium, we have to introduce the concept of a mixed strategy.

A mixed strategy $\sigma^i$ of player i is a probability distribution $(p^i_1,\ldots)$ over $S^i$ such that the kth component of $(p^i_1,\ldots)$ is the probability with which i randomly plays her kth pure strategy in $S^i$. The set of all i's mixed strategies is denoted $\Sigma^i$. The Cartesian product $\Sigma = \times_i\Sigma^i$ is called the mixed strategy profile set, of which each element is called a mixed strategy profile, consisting a mixed strategy from each and every player. Given a mixed strategy profile $\sigma$, the expected payoff for a player i is a convex combination of all possible payoffs to i: $\{\pi^i(s) : s\in S\}$, assuming that all the players randomly and independently choose pure strategies with the specified probabilities defined by their mixed strategies. The coefficient for each payoff equals to the probability for the corresponding pure strategy profile to be played. For convenience we denote by $\pi^i(\sigma)$ the expected payoff to i when $\sigma$ is played, although the mapping $\pi^i$ now has changed its meaning.

In extensive form games, it is more convenient to consider "behavior strategies" than the mixed strategies defined in the last paragraph. A behavior strategy $\sigma^i$ of Player i specifies a probability distribution on $M(X^i_j)$ for i's every information set $X^i_j$. Thus, the kth component of $\sigma^i(X^i_j)$ is the probability for i to play the kth move in $M(X^i_j)$. In the following game tree, player 2 has two information sets, and in each of them she has two choices of moves. Thus she has four pure strategies: {ac, ad, bc, bd}. A mixed strategy of her looks like (p,q,r,1-p-q-r). On the other hand, a behavior strategy looks like ((x,1-x),(y,1-y)). Her mixed strategy set is three-dimensional, and her behavior strategy set is only 2-dimensional. Given a mixed strategy (p,q,r,1-p-q-r), when she reaches the upper information set, the probability with which she plays a is Prob(a) = Prob(ac)+Prob(ad) = p+q; when she reaches the lower information set, the probability with which she plays c is p+r. As a result, the mixed strategy (p,q,r,1-p-q-r) leads to the behavior strategy ((p+q,1-p-q)(p+r,1-p-r)). Conversely, given a behavior strategy ((x,1-x),(y,1-y)), to find a corresponding mixed strategy, we need to solve: p+q=x, p+r=y. We have (usually infinitely many) solutions: r=t, q=t+(x-y), p=-t+y, for any t satisfying $\max\{0,y-x\} \le t \le \min\{y,1-x\}$.




For any extensive form game with perfect recall (a game with every player in any stage being able to recall all the information he has received so far), any mixed strategy can be "reduced" to an equivalent behavior strategy, and for any behavior strategy, there exists a set of mixed strategies equivalent to it.

2. A Nash equilibrium in mixed strategies of a game G is a strategy profile $\sigma^* = \langle\sigma^{*1}, ..., \sigma^{*I}\rangle$, such that for every i,

$$\pi^i(\sigma^*) \geq \pi^i(\sigma^*|\sigma^i), \quad \forall\sigma^i \in \Sigma^i$$

where $\sigma^*|\sigma^i$ is the strategy profile obtained from $\sigma^*$ by replacing $\sigma^{*i}$ with $\sigma^i$. A game may or may not have a Nash equilibrium in pure strategies. However, John Nash was able to prove that every finite game in strategic form has at least one Nash equilibrium in mixed strategies.

3. **Theorem 1**. (The Nash Theorem) Any finite game $[P,S,\pi]$ has at least one Nash equilibrium in mixed strategies.

Proof. We will define a correspondence on $\Sigma$. For any given $\sigma \in \Sigma$, we write $\sigma^{-i} = \langle\sigma^1, ..., \sigma^{i-1}, \ , \sigma^{i+1}, ..., \sigma^I\rangle$. Let $\tau^i$ be a best response of Player i to $\sigma^{-i}$:

$$\pi^i(\langle\sigma^1, ..., \sigma^{i-1}, \tau^i, \sigma^{i+1}, ..., \sigma^I\rangle) \geq \pi^i(\langle\sigma^1, ..., \sigma^{i-1}, \omega^i, \sigma^{i+1}, ..., \sigma^I\rangle), \forall \omega^i \in \Sigma^i$$

Such a $\tau^i$ must exists because of the continuity of $\pi^i(\langle\sigma^1, ..., \sigma^{i-1}, \cdot, \sigma^{i+1}, ..., \sigma^I\rangle)$ in $\Sigma^i$. Let $B^i = \{\tau^i \in \Sigma^i : \tau^i$ is a best response of Player i to $\sigma^{-i}\}$. Then $B^i$ is closed in $\Sigma^i$ and convex. In fact, the closedness of $B^i$ follows from the continuity of $\pi^i(\langle\sigma^1, ..., \sigma^{i-1}, \cdot, \sigma^{i+1}, ..., \sigma^I\rangle)$, and the convexity of $B^i$ follows from the the property that $\pi^i(\langle\sigma^1, ..., \sigma^{i-1}, \lambda\tau^i + (1-\lambda)\underline{\tau}^i, \ , \sigma^{i+1}, ..., \sigma^I\rangle) = \lambda\pi^i(\langle\sigma^1, ..., \sigma^{i-1}, \tau^i, \sigma^{i+1}, ..., \sigma^I\rangle) + (1-\lambda)\pi^i(\langle\sigma^1, ..., \sigma^{i-1}, \underline{\tau}^i, \sigma^{i+1}, ..., \sigma^I\rangle)$. Let $B = B^1 \times ... B^I$. Then B is closed in $\Sigma$ and convex. Now we have a point-to-set mapping $\Phi$ defined on the convex and compact set $\Sigma$ such that $\Phi(\sigma) = B$. To show that $\Phi$ has a fixed point, we need to show that $\Phi$ is upper semi-continuous. Let $\{\sigma_k\}$ be a sequence of strategy profiles contained in $\Sigma$ with $\lim \sigma_k = \sigma$. Let $\tau_k \in B_k = \Phi(\sigma_k)$ with $\lim \tau_k = \tau$. We want to show that $\tau \in \Phi(\sigma)$. If not, there must exists some i such that $\tau^i$ is not a best response of Player i to $\sigma^{-i}$. Let $\omega^i$ be the best response of Player i to $\sigma^{-i}$. Then

$$\pi^i(\langle\sigma^1, ..., \sigma^{i-1}, \tau^i, \sigma^{i+1}, ..., \sigma^I\rangle) = \pi^i(\langle\sigma^1, ..., \sigma^{i-1}, \omega^i, \sigma^{i+1}, ..., \sigma^I\rangle) - d$$

with $d > 0$. By the continuity of $p^i$ on S, for k sufficiently large we have

$$\pi^i(\langle\sigma_k^1, ..., \sigma_k^{i-1}, \tau_k^i, \sigma_k^{i+1}, ..., \sigma_k^I\rangle) < \pi^i(\langle\sigma_k^1, ..., \sigma_k^{i-1}, \omega_k^i, \sigma_k^{i+1}, ..., \sigma_k^I\rangle) - d/2$$

it gives a contradiction to $\tau_k \in B_k$. Thus $\tau \in \Phi(\sigma)$, and $\Phi$ is thus upper semi-continuous. By Kakutani's Fixed Point Theorem there must exist some $\sigma^*$ with $\sigma^* \in \Phi(\sigma^*)$. It is easy to see that $\sigma^*$ is a Nash equilibrium of the game $[P,S,\pi]$.

Remark: The compactness of the mixed strategy profile set $\Sigma$ is important to guarantee the NE existence. In Example 2 given below, one can see that some infinite game may have no Nash equlibrium because $\Sigma$ is not compact.

Since every finite game in extensive form with perfect recall can be translated into a finite game in strategic form, this Nash Theorem also applies to finite extensive form games with perfect recall:

**Corollary 1**. Any finite extensive form game with perfect recall has at least one Nash equilibrium in behavior strategies.

4. For the computation of a mixed strategy Nash equilibrium, we need the following

proposition.

**Proposition 1**. Assume that $<\sigma^i, \sigma^{-i}>$ is a Nash equilibrium of G and $\sigma^i = (p^i_1, ..., p^i_K)$. Then, for any $p^i_h, p^i_k > 0$, we must have

$$\pi^i(<s^i_h, \sigma^{-i}>) = \pi^i(<s^i_k, \sigma^{-i}>) = \pi^i(<\sigma^i, \sigma^{-i}>)$$

In another word, at any NE, if a player plays a mixed strategy $\sigma^i$, then he would get the same (expected) payoff if he had played any pure strategy which appears in $\sigma^i$ with a positive probability. We thus say that, $\sigma^{-i}$ is equalizing against i's every pure strategy adopted by i with a positive probability.

We leave the proof of this proposition as an exercise.

5. **Example 1**. (Stone-Paper-Scissors Game) Two children, 1 and 2, play the Stone-Paper-Scissors Game. They agree with the payoff bimatrix given on the next page. Compute all possible Nash equilibria.

| | | Player 2 | | |
|---|---|---|---|---|
| | | stone | paper | scissors |
| | Stone | 0, 0 | -1, 1 | 2, -2 |
| Player 1 | Paper | 1, -1 | 0, 0 | -3, 3 |
| | Scissors | -2, 2 | 3, -3 | 0, 0 |

[Analysis]. We first argue that there is no NE with anyone playing any pure strategy. By symmetry we need only show that there is no NE with 1 playing Stone. Actually if 1 does play Stone, the only best response by 2 is paper; but when paper is played by 2, the best response of 1 is not Stone but Scissors. Thus Player 1 playing Stone can never be part of any Nash equilibrium. In general there is no NE with anyone playing any pure strategy.

Now we argue that there is no NE with any player playing a two-way mixed strategy. By symmetry, we need only show that Player 1 playing a mixed strategy of Stone and Paper can never be part of a Nash equilibrium. In fact, in 1 does play in this way, any best response of 2 will never contain the playing of stone, since it is strictly dominated by paper. But when stone is never played by 2, 1's playing of Paper is not rational since it is strictly dominated by Scissors. Thus with 1 playing a two way mixed strategy of Stone and Paper, no NE can be achieved. In general there is no NE with any player playing a two-way mixed strategy.

Now we look for an NE with each player playing a three-way mixed strategy. Let an NE strategy for Player 1 being $(p,q,r)$ with all entry positive and the sum of them being 1. Because 2's best response is also three-way mixed, according to Proposition 1, the expected payoff to 2 when she plays each of the pure strategy must always be the same. (i.e. $(p,q,r)$ is equalizing against 2's every pure strategy.) Thus we have

$0p - 1q + 2r = 1p + 0q - 3r = -2p + 3q + 0r\,; \quad p + q + r = 1$

From the above system one can easily solve $p = 1/2$, $q = 1/3$ and $r = 1/6$. By symmetry the unique NE of this game is <(1/2,1/3,1/6),(1/2,1/3,1/6)>. The corresponding expected payoff vector is (0,0).

6. Though we have the NE existence theorem for all finite games in normal form, for |P| > 2, in the general case, there is no algorithm for computing an NE. On the other hand, for bimatrix games, the nonlinear programming method can be used for NE computation. But we will not discuss it in this course.

7. We now look at an example without NE existence.

**Example 2.** Each of two players announces a natural number simultaneously. The one giving the smaller number pays 1 dollar to his opponent. No payment is required in case the two numbers are equal.

[Analysis]. Now the pure strategy set $S^i$ consists of a countably infinite pure strategies. A mixed strategy of i is a sequence of nonnegative real numbers
$x^i_1, x^i_2, \ldots, x^i_n, \ldots$, such that $\sum_{n=1}^{\infty} x^i_n = 1$. Here $x^i_n$ is the probability for i to announce the natural number n.

We will show that no strategy profile can be a Nash equilibrium. Consider any strategy profile $<(x^1_1, x^1_2, \ldots, x^1_n, \ldots),(x^2_1, x^2_2, \ldots, x^2_n, \ldots)>$. Without of loss of generality we may assume the associated (expected) payoff to 1 is not greater than 0. We will show that 1 can make an improvement.

Because of the fact $\sum_{n=1}^{\infty} x^2_n = 1$. There must exists some N sufficiently large, such that $\sum_{n=1}^{N} x^2_n > 2/3$. Imagine that Player 1 now changes his strategy to a pure strategy of announcing N+1. Then his expected payoff is

$$\sum_{n=1}^{N} x^2_n - \sum_{n=N+2}^{\infty} x^2_n > 2/3 - 1/3 = 1/3.$$

i.e. 1 can really make an improvement. Therefore the given strategy profile is not an NE.

8. **Exercise**. Construct a 3×3 bimatrix game which has precisely seven Nash equilibria.

9. **Exercise**. Construct a finte extensive form game with perfect information which has at least four Nash equilibria and has but a unique subgame perfect equilibrium.